Received: *April 16, 1999*, Accepted: *May 24, 1999*

Solitons on the supermembrane

Eric Bergshoeff

Institute for Theoretical Physics
Nijenborgh 4, 9747 AG Groningen, The Netherlands
E-mail: E.A.Bergshoeff@phys.rug.nl

Paul K. Townsend

DAMTP, Silver Street, University of Cambridge,
Cambridge CB3 9EW, United Kingdom
E-mail: P.K.Townsend@damtp.cam.ac.uk

Abstract: Energy bounds are derived for planar and compactified M2-branes in a hyper-Kähler background. These bounds are saturated, respectively, by lump and Q-kink solitons, which are shown to preserve half the worldvolume supersymmetry. The Q-kinks have a dual IIB interpretation as strings that migrate between fivebranes.

Keywords: M-Theory, p-branes, Brane Dynamics in Gauge Theories.

JHEP05(1999)021

Contents

1. Introduction

Supersymmetric sigma models in 2+1 dimensions with a Kähler target space generally admit static soliton-like 'lump' solutions with energy $E = |T|$, where T is the topological charge $\int \omega$ obtained by integrating the Kähler 2-form ω over the image in target space of the 2-dimensional space (see e.g. [1]). If the Kähler target space admits a holomorphic Killing vector field k then one can perform a 'Scherk-Schwarz' (SS) dimensional reduction to arrive at a 'massive' supersymmetric sigma model in 1+1 dimensions with a scalar potential $V \sim k^2$. This theory admits 'Q-kink' solutions [2, 3] with an energy

$$E = \sqrt{Q_0^2 + Q^2}\,, \tag{1.1}$$

where Q_0 is the Noether charge associated with k, and $Q = \int i_k\omega$, the integral being taken over the image in target space of the 1-dimensional space. Because k is holomorphic the 1-form $i_k\omega$ is closed, so Q is a topological charge. When $Q_0 \neq 0$ the Q-kink is a *time-dependent* solution of the sigma-model field equations. When $Q_0 = 0$ it becomes a standard static kink solution.

A 2+1 dimensional supersymmetric sigma model with a Kähler target space has an N=2 supersymmetry and the topological charge T appears as a central charge in the supersymmetry algebra. This implies the bound $E \geq |T|$, which is saturated by the sigma-model lumps. Similarly, 1+1 dimensional massive supersymmetric sigma models obtained by SS dimensional reduction actually have (2,2) supersymmetry, and both Q_0 and Q appear in the supersymmetry algebra as central charges. This implies the bound $E \geq \sqrt{Q_0^2 + Q^2}$, which is saturated by the Q-kinks.

If the Kähler target space is actually hyper-Kähler then the topological charge T of the 2+1 dimensional model is just one of a triplet of topological charges

$$\mathbf{T} = \int \boldsymbol{\omega}\,, \tag{1.2}$$

where $\boldsymbol{\omega}$ is the triplet of Kähler 2-forms. The number of supersymmetries is also doubled to N=4, and the triplet of charges $\mathbf{T}$ appear as central charges in the N=4 supersymmetry algebra. If the hyper-Kähler space admits a tri-holomorphic Killing vector field k then SS dimensional reduction along its orbits yields a (4,4) supersymmetric massive sigma model in 1+1 dimensions, again with $V \sim k^2$. The topological charge Q is now one of a triplet of topological charges

$$\mathbf{Q} = \int i_k \boldsymbol{\omega}\,, \tag{1.3}$$

and the four charges $(Q_0, \mathbf{Q})$ appear as central charges in the (4,4) supersymmetry algebra. This implies the bound

$$E \geq \sqrt{Q_0^2 + \mathbf{Q}\cdot\mathbf{Q}}\,, \tag{1.4}$$

which is saturated by the (hyper-Kähler) Q-kinks.

There is a close analogy here to N=2 and N=4 supersymmetric Yang-Mills (SYM) theories in 4+1 and 3+1 dimensions [2, 4]. The lumps of the 2+1 dimensional sigma model are similar to the instantonic solitons of the 4+1 SYM theory; for example, they have no fixed scale. The Q-kinks of the 1+1 dimensional sigma model are similar to the dyons of 3+1 SYM theory; for example the sigma model has a vacuum angle and Q-kinks generally have fractional Q_0-charge, just as SYM dyons generally carry fractional electric charge for nonzero vacuum angle. The scale introduced by the potential term in the 1+1 dimensional sigma model is analogous to the scale introduced by the Higgs mechanism in the SYM case.

The N=2 and N=4 SYM theories have interpretations in IIB string theory as effective field theories describing the fluctuations of D-branes around some 'vacuum' brane configuration. The dyon solutions are the field theory realization of (p,q) strings, or string webs, stretched between the D-branes. A feature of the brane interpretation of the SYM theories is that in a limit in which the individual branes become widely separated the dyon solutions must transmute into a solution of the equations governing the dynamics of a *single* brane. This is an Abelian SYM theory, although not of a conventional type because the brane action involves higher derivative interactions. These 'DBI solitons', were found in [5, 6]; the supersymmetric solutions are worldvolume 'spikes' of infinite total energy per unit length equal to the tension of a (p,q) string. Solutions with finite *total* energy can be found by considering the DBI action in an appropriate supergravity background [7].

These considerations motivate us to seek an interpretation of sigma-model lumps and Q-kinks as solitons on the worldvolume of the eleven-dimensional supermembrane [8], otherwise known as the M2-brane. An M2-brane in a vacuum background has supersymmetric, but infinite energy, vortex solutions that can be interpreted as intersections with other M2-branes [5, 6, 9]. In a non-vacuum Kähler background we may have the option of wrapping the 'other' M2-branes on finite area holomorphic 2-cycles of the background. These are finite energy solitons that provide the brane realization of Kähler sigma-model lumps. We shall concentrate here on the hyper-Kähler case; specifically, we shall consider the supermembrane in a background for which the 4-form field-strength vanishes and the 11-metric takes the form

$$ds^2 = ds^2(\mathbb{E}^{(1,5)} \times S^1) + ds_4^2\,, \tag{1.5}$$

where ds_4^2 is the Kaluza-Klein (KK) monopole metric

$$ds_4^2 = V^{-1}\,(d\varphi - \mathbf{A})^2 + V\,ds^2(\mathbb{E}^3)\,. \tag{1.6}$$

The 1-form $\mathbf{A}$ on $\mathbb{E}^3$ satisfies $\boldsymbol{\nabla} V = \boldsymbol{\nabla} \times \mathbf{A}$, which implies that V is harmonic on $\mathbb{E}^3$. The vector field $\partial/\partial\varphi$ is Killing and triholomorphic. We take it to be the vector field k of the previous discussion, i.e.

$$k = \frac{\partial}{\partial\varphi}\,. \tag{1.7}$$

The orbits of k are Kaluza-Klein (KK) circles which shrink to points at singularities of V. Let $\mathbf{X}$ be Cartesian coordinates on $\mathbb{E}^3$ and $\mathbf{X}_0$ a constant 3-vector from the origin. The simplest choice of V that serves our purposes is

$$V = 1 + \frac{1}{|\mathbf{X} + \mathbf{X}_0|} + \frac{1}{|\mathbf{X} - \mathbf{X}_0|}\,, \tag{1.8}$$

which describes a two-centre KK-monopole of M-theory.

Upon reduction on orbits of k, the KK monopole acquires an interpretation as two parallel IIA D6-branes separated in $\mathbb{E}^3$ by the constant vector $2\mathbf{X}_0$. The two centres of the metric at $\mathbf{X} = \pm\mathbf{X}_0$ can be considered as the poles of a 2-sphere parametrized by φ and the distance from one D6-brane along the line joining the two of them. A membrane wrapped on this 2-sphere has a IIA interpretation as a string stretched between the two D6-branes [10]. Now consider a D2-brane parallel to the two D6-branes. In general it will not be colinear in $\mathbb{E}^3$ with the two D6-branes and so will not intersect the string joining them. However, we may move it until it does intersect. From the D=11 perspective we then have a pointlike intersection of two M2-branes, one an infinite planar one and the other one wrapped on a finite area 2-cycle of the background. The singular intersection point may be desingularized so that we have a single M2-brane with a non-singular lump soliton on it of some finite size L. From the IIA perspective this corresponds to separating the points at which the strings from each of the two D6-branes meet the D2-brane.

In the case of the lump, the vacuum is an infinite planar M2-brane. To find a brane interpretation of the hyper-Kähler Q-kink we will need to wrap this M2-brane on some one-cycle of the background space. This corresponds to SS reduction on some Killing vector field with closed orbits. The dimensional reduction will preserve all supersymmetries only if this Killing vector field is triholomorphic. The Killing vector field k of (1.7) is therefore an obvious candidate, but SS reduction on orbits of k does *not* yield a potential $V \sim k^2$ as one might have expected from our earlier summary of the results of SS reduction in sigma models. Rather, it yields a non-vanishing, and non-uniform, IIA string tension. The non-uniformity of the tension creates an attractive force between the string and the D6-brane but on reaching the D6-brane core the string can simply dissolve into Born-Infeld flux. To get the potential term in the dimensionally reduced action one must suppose that the 11-metric (1.5) has another tri-holomorphic Killing vector field with closed orbits. We may take this to be a vector field generating the U(1) isometry of the S^1 factor in this metric. Let us call this vector field ℓ. Dimensional reduction on orbits of $k + \ell$ leads to a bound state of the IIA string discussed above with a D2-brane wrapped on orbits of ℓ. This bound state is itself bound to the D6-brane. The effective string action is the desired brane version of the massive hyper-Kähler sigma model, admitting Q-kink solutions. T-dualizing in the (compact) ℓ direction yields a (1,1) IIB string bound to a D5-brane. As we shall see, the Q-kink solution can then be interpreted as a (1,1) string that migrates from one D5-brane to another.

Although lump and Q-kink solutions are known to minimise the energy of the relevant sigma model it does not immediately follow that they minimise the energy on the M2-brane because of the nonlinearities of the Dirac membrane action. By means of the brane version of the Bogomol'nyi argument [9], we show that the energy of the M2-brane is indeed minimised by these solutions. We consider the lumps first, as these are static, and then generalize to the Q-kinks. Both configurations are then shown to preserve some fraction of the worldvolume supersymmetry. Again, this is known in the sigma-model case, but the supersymmetry transformations of the supermembrane are different. They can be deduced from a combination of the target space supersymmetry and the kappa-symmetry of the supermembrane, and this leads to a simple condition for a worldvolume field configuration to preserve some fraction of supersymmetry [11, 12, 13]. For a vacuum background this condition is easily interpreted as a constraint on the 32 independent constant Killing spinors of the background, but its interpretation is less direct in a non-vacuum background in which the Killing spinors are not constant and span a space of lower dimension. Here we present a more geometrical derivation of the conditions for preservation of supersymmetry and we discuss some subtleties of the non-vacuum case that have been passed over previously.

2. Energy bounds

Our starting point for finding soliton solutions as minimum energy configurations of the supermembrane will be its Hamiltonian formulation [14]. Let $\xi^i = (t, \sigma^a)$ be the worldvolume coordinates, with σ^a the worldspace coordinates, and let X^m be the D=11 spacetime coordinates. The supermembrane Lagrangian, omitting fermions, can then be written as

$$\mathcal{L} = P_m \dot{X}^m - s^a P_m \partial_a X^m - \frac{1}{2} v \left[P^2 + \det(g_{ab}) \right] , \tag{2.1}$$

where

$$g_{ab} = \partial_a X^m \partial_b X^n g_{mn} \tag{2.2}$$

is the induced worldspace metric, P_m is the 11-momentum conjugate to X^m, and s^a and v are Lagrange multipliers. Let $X^m = (Y^i, X^I)$ $(i = 0, 1, 2)$ so that

$$ds^2_{11} = dY^i dY^j \eta_{ij} + dX^I dX^J g_{IJ} , \tag{2.3}$$

where η is the 3-dimensional Minkowski metric. We make the gauge choice $Y^i(\xi) = \xi^i$. This implies that

$$g_{ab} = \eta_{ab} + \partial_a X^I \partial_b X^J g_{IJ} . \tag{2.4}$$

It also implies that

$$P_m = \left(-\varepsilon - 1, -P_I \partial_a X^I, P_I \right) , \tag{2.5}$$

where ε is the energy density relative to the brane vacuum (which is taken to have unit tension). The Hamiltonian constraint imposed by v can be solved for ε

$$\begin{aligned} (\varepsilon + 1)^2 = {} & 1 + \nabla X^I \cdot \nabla X^J g_{IJ} + (g^{IJ} + \nabla X^I \cdot \nabla X^J) P_I P_J \\ & + \frac{1}{2} (\nabla X^I \times \nabla X^J)(\nabla X^K \times \nabla X^L) g_{IK} g_{JL} , \end{aligned} \tag{2.6}$$

where we have used standard 2D vector calculus notation for worldspace derivatives. This expression differs in several respects from the corresponding expression for the sigma-model energy density. Firstly, the supermembrane expression is quadratic in ε; this is because the sigma-model approximation is a kind of non-relativistic approximation to the supermembrane (they differ in the same way that the energies of a relativistic and non-relativistic particles differ). Secondly, the supermembrane expression involves terms quartic in derivatives that are absent in the sigma-model case.

2.1 Lumps

We now aim to rewrite the above expression for the energy density in the form

$$\begin{aligned} (\varepsilon + 1)^2 = {} & \left(1 \pm \frac{1}{2} \nabla X^I \times \nabla X^J \omega_{IJ} \right)^2 + \\ & + \frac{1}{2} \left(\nabla X^I \pm * \nabla X^K I_K{}^I \right) \left(\nabla X^J \pm * \nabla X^L I_L{}^J \right) g_{IJ} + \\ & + \frac{1}{4} \sum_{r=1}^{6} \left(\nabla X^I \times \nabla X^J \Omega^{(r)}_{IJ} \right)^2 , \end{aligned} \tag{2.7}$$

where we have set $P_I = 0$ and $*\nabla = (\partial_2, -\partial_1)$ if $\nabla = (\partial_1, \partial_2)$. We assume that $I_I{}^J$ is a complex structure, that the 8-metric g_{IJ} is Hermitian with respect to it and that $\omega_{IJ} = I_I{}^K g_{KJ}$ is the corresponding closed Kähler 2-form. For the moment we leave unspecified the six 2-forms $\Omega^{(r)}$. These conditions are already sufficient to ensure that all but the quartic terms in ∇X of (2.6) are reproduced. To reproduce the quartic terms too we require that

$$X^{IJ}X^{KL}\left[\omega_{IJ}\omega_{KL} + \sum_{r=1}^{6}\Omega^{(r)}_{IJ}\Omega^{(r)}_{KL} - 2g_{KI}g_{JL}\right] = 0\,, \tag{2.8}$$

where

$$X^{IJ} \equiv \nabla X^I \times \nabla X^J\,. \tag{2.9}$$

Note that X^{IJ} is an antisymmetric 8×8 matrix. If none of its 4 skew-eigenvalues vanish, then (2.8) implies that

$$\omega_{I(J}\omega_{K)L} + \sum_{r=1}^{6}\Omega^{(r)}_{I(J}\Omega^{(r)}_{K)L} = g_{I(K}g_{J)L} - g_{KJ}g_{IL}\,. \tag{2.10}$$

For a membrane in flat space this condition is satisfied by taking the matrices

$$I_I{}^J \equiv \omega_{IK}g^{KJ}\,, \qquad (J^{(r)})_I{}^J \equiv \Omega^{(r)}_{IK}g^{KJ} \tag{2.11}$$

to be the seven complex structures of $\mathbb{E}^8$.

For every vanishing skew eigenvalue of X^{IJ} the dimension of the transverse space is effectively reduced by two. In this reduced space, we must again have (2.10) but it may now be possible to choose some of the six J matrices to vanish. For example, if X^{IJ} has two vanishing skew-eigenvalues then the transverse space is effectively 4-dimensional; in other words, there are four 'active scalars'. We may now set all but two of the J matrices to zero. The other two, together with I can be taken to be the three almost complex structures of the transverse 4-manifold (these will be covariantly constant if this transverse 4-space is hyper-Kähler, but we need not assume any special properties at this point). If X^{IJ} has three vanishing skew-eigenvalues, corresponding to two active scalars, then the transverse space is effectively two-dimensional, and we may take all the J matrices to vanish.

Given (2.8) we deduce that

$$\varepsilon \geq \frac{1}{2}|X^{IJ}\omega_{IJ}| \tag{2.12}$$

with equality when

$$\nabla X^I = \mp * \nabla X^J I_J{}^I \tag{2.13}$$

and

$$X^{IJ}\Omega^{(r)}_{IJ} = 0\,, r = 1, \ldots, 6\,. \tag{2.14}$$

The condition (2.13) is the statement that in complex coordinates Z^α, adapted to the complex structure I, the functions $Z^\alpha(z)$ are holomorphic on worldspace, with

$z = \sigma^1 + i\sigma^2$. The conditions (2.14) are implied by (2.13) if the matrices $J^{(r)}$ are such that

$$IJ^{(r)} + J^{(r)}I = 0\,, \qquad\qquad r = 1, \dots, 6\,. \tag{2.15}$$

This is true when $I, J^{(r)}$ are the 7 complex structures of $\mathbb{E}^8$. It is also satisfied if $I, J^{(1)}, J^{(2)}$ are the three almost complex structures of a 4-dimensional space, with the other J matrices vanishing. This is the case of most interest here because we may obviously reduce the transverse 8-space to an effective transverse 4-space by requiring all scalars to vanish except those associated with the ds_4^2 metric in (1.5). This restriction still allows configurations with either two or four active scalars.

In the case of a flat background, a solution of (2.13) with $2n$ real 'active scalars' has the interpretation as the (orthogonal) intersection with the worldvolume of n M2-branes, corresponding to a spacetime intersection of $n+1$ M2-branes. The spacetime configuration is known to preserve the fraction $1/2^{n+1}$ of the spacetime supersymmetry [15] so we may expect the fraction of worldvolume supersymmetry preserved to be $1/2^n$. This can be confirmed directly from a consideration of κ-symmetry of the supermembrane [16, 17]. The lump solution of (2.13) for the KK-monopole background is also one with two 'active scalars' and preserves half the worldvolume supersymmetry but the total number of worldvolume supersymmetries is half what it would be in a flat spacetime. The fraction of supersymmetry of the M-theory vacuum that is preserved by the total system is therefore 1/8 (1/2 for the solution, 1/2 for the brane and 1/2 for the background). We shall examine the question of supersymmetry in more detail in section 3.

2.2 Q-Kinks

We now set

$$\partial_2 X^I = k^I\,, \tag{2.16}$$

where k is a holomorphic Killing vector field. The holomorphicity condition ensures that the dimensionally reduced 1+1 dimensional theory preserves the $N=2$ supersymmetry of the (2+1)-dimensional model. Any additional supersymmetries will be associated with additional complex structures; if k is holomorphic with respect to them too then the reduction will preserve these additional supersymmetries. For the KK-monopole background we may take k to be the triholomorphic Killing vector of (1.7). Using (2.16) in (2.6) we have:

$$\begin{aligned}(\varepsilon+1)^2 = 1 &+ \left(g^{IJ} + \partial X^I \partial X^J + k^I k^J\right) P_I P_J + |\partial X|^2 + |k|^2 + \\ &+ 2\,\partial X^{[I} k^{J]} \partial X^{[K} k^{L]} g_{IK} g_{JL}\,,\end{aligned} \tag{2.17}$$

where $\partial X = \partial_1 X$. Restricting to static ($P=0$) and uniform ($\partial X = 0$) configurations yields $\epsilon = \sqrt{1+|k|^2} - 1 \approx \frac{1}{2}|k|^2$, which is the membrane version of the scalar potential that leads to Q-kink solutions interpolating between its minima at fixed points of k where $|k|$ vanishes.

Under the same conditions as before, the expression (2.17) for the energy density can be rewritten as

$$
\begin{aligned}
(\varepsilon+1)^2 = {} & \left[1 + vk\cdot P + \sqrt{1-v^2}\;\partial X^I k^J \omega_{IJ}\right]^2 + |P - vk|^2 + \\
& + \left|\partial X^I + \sqrt{1-v^2}\; k^J I_J{}^I\right|^2 + (P\cdot\partial X)^2 + \\
& + \left[v\;\partial X^I k^J \omega_{IJ} - \sqrt{1-v^2}\; k\cdot P\right]^2 + \sum_r \left(\partial X^I k^J \Omega^{(r)}_{IJ}\right)^2 .
\end{aligned} \tag{2.18}
$$

for arbitrary constant v with $|v| < 1$. We deduce that

$$
\varepsilon \geq vk\cdot P + \sqrt{1-v^2}\;\partial X^I k^J \omega_{IJ}\,, \tag{2.19}
$$

with equality when

$$
P^I = vk^I\,, \tag{2.20}
$$

$$
\partial X^I = -\sqrt{1-v^2}\; k^J I_J{}^I\,, \tag{2.21}
$$

since these equations imply the vanishing of the remaining terms.

Setting $P^I = \dot{X}^I$ in (2.20) we recover the equations found in [2], the solutions of which are Q-kinks. The explicit Q-kink solution of [2] was given for the two-centre metric with V as in (1.8) but without the constant term (i.e. for the Eguchi-Hanson metric [18]). The explicit solution when V includes a constant term has been found by Opfermann [19].

3. Supersymmetry

The supermembrane is invariant under all isometries of the background. Supersymmetries correspond to Grassmann odd Killing vector superfields $\chi = \chi^A E_A$, where $E_A = E_A{}^M \partial_M$. The (Grassman even) spinor component χ^α is a Killing spinor in the standard sense, at least in a purely bosonic background. The (Grassman odd) vector component χ^a is a superfield satisfying the constraint $\mathcal{D}_\alpha \chi^a = (\Gamma^a\chi)_\alpha$. Let $\{\chi\}$ be the complete set of these Killing vector superfields and let $\{\epsilon\}$ be a corresponding set of anticommuting parameters. The supersymmetry transformations of the worldvolume fields Z^M are then

$$
\delta_\epsilon Z^M = \epsilon\cdot\chi^M\,, \tag{3.1}
$$

where $\epsilon\cdot\chi$ is used to denote the sum over the (ϵ,χ) pairs. Defining $\delta E^A = \delta Z^M E_M{}^A$, we then have

$$
\delta_\epsilon E^A = \epsilon\cdot\chi^A\,. \tag{3.2}
$$

The κ-symmetry variation $\delta_\kappa Z^M$ can be similarly expressed in the form

$$
\delta_\kappa E^\alpha = \kappa^\beta (1+\Gamma)_\beta{}^\alpha\,, \qquad \delta_\kappa E^a = 0\,, \tag{3.3}
$$

where

$$\Gamma = \frac{1}{6\sqrt{-g}} \epsilon^{ijk} E_i{}^a E_j{}^b E_k{}^c \Gamma_{abc} \,, \tag{3.4}$$

with $E_i{}^a = \partial_i Z^M E_M{}^a$, and g is the determinant of the induced worldvolume metric $g_{ij} = E_i{}^a E_j{}^b \eta_{ab}$. To fix κ-symmetry, we make the gauge choice [11]

$$E^\alpha (1+\Gamma)_\alpha{}^\beta = 0 \,. \tag{3.5}$$

This restricts only dZ^M, but this is sufficient. Note that this gauge choice is invariant under supersymmetry, at least in a bosonic background and for vanishing worldvolume fermions; under these conditions we may neglect the variation of Γ, while

$$\delta_\epsilon (dZ^M E_M{}^\alpha) = D(\epsilon \cdot \chi)^\alpha - \epsilon \cdot \chi^\beta E^\gamma T_{\gamma\beta}{}^\alpha \,, \tag{3.6}$$

which vanishes by the Killing spinor equation (the $T_{\gamma\beta}{}^\alpha$ component of the torsion tensor is proportional to the 4-form field strength of D=11 supergravity).

The remaining physical variables are such that their variations are $\delta E^\alpha (1-\Gamma)_\alpha{}^\beta$. The condition that the worldvolume configuration preserves some supersymmetry is therefore

$$\epsilon \cdot \chi^\alpha (1-\Gamma)_\alpha{}^\beta = 0 \,. \tag{3.7}$$

For flat superspace, $\chi_I{}^\alpha = \delta_I{}^\alpha$, so $\epsilon \cdot \chi = \epsilon$, a constant 32-component spinor. We thus recover the flat space condition [11]

$$\epsilon^\alpha (1-\Gamma)_\alpha{}^\beta = 0 \,. \tag{3.8}$$

More generally, we must take into account the fact that $\epsilon \cdot \chi$ is neither constant nor a spinor with 32 independent components. For the simplest backgrounds, including the KK-monopole background considered here, we have

$$\epsilon \cdot \chi = f_\chi \, \epsilon \,, \tag{3.9}$$

where f_χ is an ordinary function, and ϵ is a constant 32-component spinor satisfying

$$P_\chi \, \epsilon = 0 \,, \tag{3.10}$$

with P_χ a *constant* projection matrix. For the KK-monopole background the matrix P_χ is just the product of four constant Dirac matrices, one for each of the four dimensions of the 4-metric, and it has the property (associated with the fact that this background preserves 1/2 of the spacetime supersymmetry) that $\mathrm{tr} P_\chi = 16$. The fraction of spacetime supersymmetry preserved by the brane plus background configuration is therefore determined by the number of simultaneous solutions to (3.8) and (3.10). Note that the function f_χ of (3.9) is irrelevant to the final result.

We now fix worldvolume diffeomorphisms by the 'static gauge' choice

$$X^m = \Big(\xi^i, X^I(\xi)\Big)\,. \tag{3.11}$$

With this gauge choice the condition (3.8) becomes

$$\begin{aligned}\sqrt{-g}\,\epsilon = \Big[&\Gamma_* + \Gamma^k\partial_k X^I\Gamma_I\Gamma_* + \frac{1}{2}\Gamma_k\epsilon^{ijk}\partial_i X^I\partial_j X^J\Gamma_{IJ} + \\ &+ \frac{1}{6}\varepsilon^{ijk}\partial_i X^I\partial_j X^J\partial_k X^K\Gamma_{IJK}\Big]\,\epsilon\,,\end{aligned} \tag{3.12}$$

where

$$\Gamma_* = \Gamma_{012}\,. \tag{3.13}$$

In addition

$$g = \det\left(\eta_{ij} + \tilde{g}_{ij}\right), \tag{3.14}$$

where

$$\tilde{g}_{ij} = \partial_i X^I\partial_j X^J g_{IJ}\,. \tag{3.15}$$

The condition (3.12) for preservation of supersymmetry can now be expanded in a power series in ∂X. We assume here that each term in the series must vanish separately.[1] At zeroth order in this expansion we learn that

$$\Gamma_*\epsilon = \epsilon\,. \tag{3.16}$$

Because the projector P_χ involves only the Γ^I matrices, this equation tells us that the worldvolume vacuum preserves half the supersymmetries of the supergravity background, i.e. that the M2–brane is 1/2 supersymmetric.

At first order in the ∂X expansion we learn that

$$\Gamma^k\partial_k X^I\Gamma_I\,\epsilon = 0\,. \tag{3.17}$$

This implies various higher-order identities. In particular it implies that $\det\tilde{g}_{ij}$ vanishes and that

$$\eta^{im}\eta^{jn}\tilde{g}_{jm}\tilde{g}_{in} = \frac{1}{2}(\eta^{ij}\tilde{g}_{ij})^2\,. \tag{3.18}$$

Using these identities, and the constraints on ϵ quadratic and cubic in ∂X that also follow from (3.17), one can show that the *full* constraint $\Gamma\epsilon = \epsilon$ is satisfied. Thus, (3.17) is the only condition (apart from $P_\chi\epsilon = 0$) that we need analyse to determine the fraction of supersymmetry preserved by lump and Q-kink soliton solutions.

[1] For a flat space background this amounts to the assumption that the worldspace is the contact set of a Kähler calibration. Kähler calibrations are only ones of relevance here, although for the M5-brane there are other calibrations for which the assumption would be false. See [16, 17, 20] for a discussion of calibrations in relation to branes.

Having found the conditions for partial preservation of worldvolume supersymmetry, we are now in a position to verify that the lump and Q-kink solitons are supersymmetric and to determine the fraction of supersymmetry they preserve. We need not discuss the lump and Q-kink cases separately because the formalism to follow will apply equally to both. For lumps we just set $v = 0$ while for Q-kinks we set $\partial_2 X^I = k^I$. We begin with the observation that the equations

$$\dot{X}^I = v\partial_2 X^I\,, \qquad \partial_1 X^I + \sqrt{1-v^2}\,\partial_2 X^J\, I_J{}^I = 0\,, \tag{3.19}$$

imply that

$$\tilde{g} = \tilde{g}_{22} \times \begin{pmatrix} v^2 & 0 & v \\ 0 & 1-v^2 & 0 \\ v & 0 & 1 \end{pmatrix}\,, \tag{3.20}$$

which manifestly has vanishing determinant and solves (3.18). Using (3.19) in (3.17) and $\Gamma_* \epsilon = \epsilon$, we have

$$\partial_2 X^I \Gamma^J \left(g_{IJ} + \tilde{\Gamma}\omega_{IJ}\right)\epsilon = 0\,, \tag{3.21}$$

where

$$\tilde{\Gamma} = \frac{1}{\sqrt{1-v^2}}\left(\Gamma^0 + v\Gamma^2\right)\,. \tag{3.22}$$

Note that $\tilde{\Gamma}^2 = -1$.

The matrices Γ^I are space-dependent. We can write them in terms of the constant complex matrices

$$\Gamma^\alpha = \Gamma^I e_I{}^\alpha \tag{3.23}$$

and their complex conjugates $\bar{\Gamma}^{\bar{\alpha}}$, where $e_I{}^\alpha$ is a complex target space vielbein (with complex conjugate $\bar{e}_I{}^{\bar{\alpha}}$) chosen such that

$$\{\Gamma^\alpha, \Gamma^\beta\} = 0 \qquad \{\Gamma^\alpha, \bar{\Gamma}^{\bar{\beta}}\} = \delta^{\alpha\bar{\beta}}\,. \tag{3.24}$$

Using the fact that $\omega_{\alpha\bar{\beta}} = i\delta_{\alpha\bar{\beta}}$ in this basis, we now have

$$\sum_{\alpha=\bar{\alpha}} \left[e^\alpha \bar{\Gamma}^{\bar{\alpha}} + \bar{e}^{\bar{\alpha}}\Gamma^\alpha + i(e^\alpha\bar{\Gamma}^{\bar{\alpha}} - \bar{e}^{\bar{\alpha}}\Gamma^\alpha)\tilde{\Gamma}\right]\epsilon = 0 \tag{3.25}$$

where

$$e^\alpha = \partial_2 X^I e_I{}^\alpha\,. \tag{3.26}$$

Each term in the sum must vanish separately. This leads to a set of equations, each of which can be written in the form

$$(e\bar{\Gamma} + \bar{e}\Gamma)(1 - i\tilde{\Gamma}[\Gamma, \bar{\Gamma}])\epsilon = 0\,. \tag{3.27}$$

It follows that *either* $e = 0$, which effectively requires one complex worldvolume scalar to be constant, *or* ϵ must satisfy the constraint

$$(1 - i\tilde{\Gamma}[\Gamma, \bar{\Gamma}])\epsilon = 0\,, \tag{3.28}$$

which reduces the fraction of supersymmetry preserved by two, unless it is already satisfied by virtue of the P_χ projection imposed by the background.

We briefly discussed the flat background case in section 2.1. Solutions with $2n$ active (real) scalars preserve $1/2^n$ of the worldvolume supersymmetry and hence $1/2^{n+1}$ of the spacetime supersymmetry; their spacetime interpretation is as $n+1$ intersecting M2-branes. The computation of the fraction of worldvolume supersymmetry preserved by the finite energy lumps and Q-kinks is slightly more involved because the effects of the P_χ projection must be taken into account. However this just reduces the initial number of supersymmetries by a factor of two. The M2-brane breaks half of that and the lump and Q-kink solitons halve it again, exactly as in the flat space case.

These results could be anticipated from the central charge structure of the supermembrane worldvolume superalgebra. In the KK-monopole background we would need to consider the N=4 D=3 worldvolume supersymmetry algebra. For simplicity we concentrate here on the N=8 D=3 algebra relevant to a supermembrane in a flat space background. As we are considering only scalar central charges, the supersymmetry algebra is [21]

$$\{Q_\alpha^{\tilde{I}}, Q_\beta^{\tilde{J}}\} = \delta^{\tilde{I}\tilde{J}} P_{\alpha\beta} + \varepsilon_{\alpha\beta} \tilde{Z}^{\tilde{I}\tilde{J}} , \tag{3.29}$$

where the 8 supersymmetry charges $Q^{\tilde{I}}$ transform as a chiral $SO(8)$ spinor. The antisymmetric central charge matrix $\tilde{Z}$ has four skew eigenvalues ζ_k $(k = 1, 2, 3, 4)$. The positivity of the $\{Q, Q\}$ anticommutator implies the bound

$$M^2 \geq \sup(\zeta_1, \zeta_2, \zeta_3, \zeta_4) . \tag{3.30}$$

The fraction of worldvolume supersymmetry preserved is 2^{n-5} where n is the number of factors of $\det\{Q, Q\}$ of the form $(M^2 - \zeta)^2$ that simultaneously vanish. For example, states for which all four skew eigenvalues are equal, but non-zero, preserve half of the worldvolume supersymmetry. Note that since $\tilde{I}$ is a *spinor* index the central charge $\tilde{Z}$ cannot be directly interpreted as the two-form topological charge Z associated with a membrane in a given 2-plane; the relation between the two is such that equal skew-eigenvalues of $\tilde{Z}$ corresponds to three vanishing skew-eigenvalues of Z, and vice-versa.

4. IIB interpretation of Q-kinks

In the introduction we explained briefly the IIA superstring interpretation of the supermembrane lump solutions. As mentioned there, the most natural superstring interpretation of Q-kinks is in terms of IIB superstring theory. We now return to this point.

It was implicit in our discussion of the Q-kink in section 2.2 that $\xi^2 = \rho$ is periodically identified; otherwise we do not have a genuine compactification. Since we had already made the static gauge choice $Y^2 = \rho$, it follows that we must take Y^2 to be an angular variable, i.e. the coordinate of the S^1 factor in (1.5). In fact, the Killing vector field $\partial/\partial Y^2$ can be identified as a multiple of the triholomorphic Killing vector field ℓ mentioned in the introduction. A standard dimensional reduction on orbits of the triholomorphic Killing vector field k would imply that Y^2 is the only field depending on ρ. However, the SS reduction ansatz of (2.16) means that Y^2 is not the only ρ-dependent worldvolume field. In fact, given (1.7), the condition (2.16) implies that $\partial_\rho \varphi = 1$, or $\varphi = \rho$ up to a constant. If we introduce the new coordinates

$$X^0 = \varphi - Y^2 \qquad \tilde{Y} = \frac{1}{2}\left(Y^2 + \varphi\right), \tag{4.1}$$

then the combination of the static gauge choice and the SS reduction imply that $(X^0, \mathbf{X})$ are ρ-independent while $\tilde{Y} = \rho$. In other words, we are wrapping the membrane on the $\tilde{Y}$ direction, i.e. on the $k + \ell$ cycle. We can consider this to be a non-marginal bound state of a membrane wrapped on the k cycle with one wrapped on the ℓ cycle [22]. The IIA interpretation of this bound state (with the k cycle interpreted as the KK circle) was explained briefly in the introduction: a membrane wrapped on the k cycle yields a IIA string in the D6-brane while a membrane wrapped on the ℓ cycle yields a D2-brane, so we end up with a IIA string bound to a D2-brane in a D6-brane. We now consider the IIB interpretation obtained by T-duality in the ℓ direction.

The IIB dual of a membrane wrapped once on each of the two cycles of the torus relating the IIB theory to M-theory is a (1,1) string [23]. In our case the (1,1) string is bound to the D5-brane that is the IIB dual of the KK-monopole. The binding is due to the fact that the D5-brane attracts (1,0) strings and is neutral to (0,1) strings. Thus, there is effectively a potential confining the (1,1) string to the D5-brane (as expected from the $V \sim k^2$ potential relevant to the IIA description; in fact, the potential is T-duality invariant [24]). Given sufficient energy, the (1,1) string could migrate from one D5-brane to another one at some position in the transverse 4-space specified by a 4-vector. In fact the supermembrane Q-kinks discussed earlier correspond to strings which begin on one D5-brane but then jump over to another one. The charge 4-vector $(Q_0, \mathbf{Q})$ is just the position 4-vector of the other D5-brane, as we now explain.

The triplet of Kähler 2-forms associated with the 4-metric (1.6) is

$$\boldsymbol{\omega} = (d\varphi + \mathbf{A} \cdot d\mathbf{X})d\mathbf{X} - V d\mathbf{X} \times d\mathbf{X}\,, \tag{4.2}$$

where the wedge product of forms is understood. Hence the triplet of topological charges $\mathbf{Q}$ is given by

$$\mathbf{Q} = \int i_k \boldsymbol{\omega} = \int d\mathbf{X}\,, \tag{4.3}$$

where the integral is over the (1,1) string worldspace. For V as given in (1.8), the potential k^2 has minima at $\mathbf{X} = \pm \mathbf{X}_0$, so a string that starts at one minimum and ends at the other one has a 3-vector kink charge $\mathbf{Q} = 2\mathbf{X}_0$. This is the same charge as in the IIA interpretation. However, the Noether charge in the IIA interpretation becomes a fourth topological charge in the IIB interpretation (cf. [24]). To see this it is simplest to get to the IIB theory by first compactifying on the ℓ cycle followed by

T-duality on the k cycle. This leads to the S-dual of the configuration obtained from performing these operations in the reverse order (i.e. a (1,1) string in a NS-5-brane), but the result we are aiming at is unaffected by S-duality. Having compactified on the ℓ cycle, T-duality on the k cycle takes $\dot\varphi$ to $\partial\tilde\varphi$, where $\tilde\varphi$ is the T-dual coordinate, and hence takes the Noether charge $Q_0 = \int V^{-1}\dot\varphi$ to the topological charge

$$\tilde{Q}_0 = \int d\tilde\varphi\,. \tag{4.4}$$

This result is to be expected from the fact that the transverse space of the IIB D5-brane is 4-dimensional. Thus, in the IIB theory the Q-kink charges $(Q_0, \mathbf{Q})$ become a single topological 4-vector charge $Q = (\tilde{Q}_0, \mathbf{Q})$. A configuration for which this charge is non-zero represents a (1,1) string that starts at one D5-brane and then migrates to another one positioned at some distance $|Q|$ from the first in the direction given by Q.

Acknowledgments

One of us (PKT) would like to thank Groningen university for its hospitality, and Neil Lambert and George Papadopoulos for discussions. The work of E.B. is supported by the European Commission TMR program ERBFMRX-CT96-0045, in which E.B. is associated to the University of Utrecht.

References

[1] P.J. Ruback, *Sigma model solitons and their moduli space metrics*, *Comm. Math. Phys.* **116** (1988) 645.

[2] E.R.C. Abraham and P.K. Townsend, *Q-kinks*, *Phys. Lett.* **B 291** (1992) 85; *More on Q-kinks: a (1+1)-dimensional analogue of dyons*, *Phys. Lett.* **B 295** (1992) 225.

[3] G. Papadopoulos and P.K. Townsend, *Solitons in supersymmetric sigma models with torsion*, *Nucl. Phys.* **B 444** (1995) 245 [hep-th/9501069];
G. Papadopoulos and A. Opfermann, *Solitons in (1,1) supersymmetric massive sigma models*, *Nucl. Phys.* **B 488** (1997) 177 [hep-th/9610099].

[4] N. Dorey, *The BPS spectrum of two-dimensional gauge theories with twisted mass terms*, *J. High Energy Phys.* **11** (1998) 005 [hep-th/9806056];
N. Dorey, T.J. Hollowood and D. Tong, *The BPS spectra of gauge theories in two and four dimensions*, *J. High Energy Phys.* **05** (1999) 006 [hep-th/9902134].

[5] C. Callan and J. Maldacena, *Brane dynamics from the Born-Infeld action*, *Nucl. Phys.* **B 513** (1998) 198 [hep-th/9708147].

[6] G.W. Gibbons, *Born-Infeld particles and Dirichlet p-branes*, *Nucl. Phys.* **B 514** (1998) 603 [hep-th/9709027].

[7] J.P. Gauntlett, C. Koehl, D. Mateos, P.K. Townsend and M. Zamaklar, *Finite energy Dirac-Born-Infeld monopoles and string junctions*, hep-th/9903156;
R. de Mello Koch, A. Paulin-Campbell anf J.P. Rodrigues, *Monopole dynamics in N=2 super-Yang-Mills theory from a threebrane probe*, hep-th/9903207.

[8] E. Bergshoeff, E. Sezgin and P.K. Townsend, *Supermembranes and eleven-dimensional supergravity*, *Phys. Lett.* **B 189** (1987) 75; *Properties of the eleven-dimensional supermembrane theory*, *Ann. Phys. (NY)* **185** (1988) 330.

[9] J.P. Gauntlett, J. Gomis and P.K. Townsend, *BPS bounds for worldvolume branes*, *J. High Energy Phys.* **01** (1998) 003 [hep-th/9711205].

[10] A. Sen, *A note on enhanced gauge symmetries in M- and string theory*, *J. High Energy Phys.* **09** (1997) 001 [hep-th/9707123]
P.K. Townsend, *M-theory from its superalgebra*, in *Strings, Branes and Dualities*, Cargèse lectures 1997, L. Baulieu, P. Di Francesco, M. Douglas, V. Kazakov, M. Picco and P. Windey eds., Kluwer 1999, pp. 141–177 [hep-th/9712004].

[11] E. Bergshoeff, M.J. Duff, C.N. Pope and E. Sezgin, *Supersymmetric supermembrane vacua and singletons*, *Phys. Lett.* **B 199** (1987) 69.

[12] K. Becker, M. Becker and A. Strominger, *Fivebranes, membranes and non-perturbative string theory*, *Nucl. Phys.* **B 456** (1995) 130 [hep-th/9507158].

[13] E. Bergshoeff, R. Kallosh, T. Ortín and G. Papadopoulos, *Kappa-symmetry, supersymmetry and intersecting branes*, *Nucl. Phys.* **B 502** (1997) 149 [hep-th/9705040].

[14] E. Bergshoeff, E. Sezgin and Y. Tanii, *Hamiltonian formulation of the supermembrane*, *Nucl. Phys.* **B 298** (1988) 187.

[15] G. Papadopoulos and P.K. Townsend, *Intersecting M-branes*, *Phys. Lett.* **B 380** (1996) 273 [hep-th/9603087].

[16] G.W. Gibbons and G. Papadopoulos, *Calibrations and Intersecting Branes*, *Comm. Math. Phys.* **202** (1999) 593 [hep-th/9803163].

[17] J.P. Gauntlett, N.D. Lambert and P.C. West, *Branes and Calibrated Geometries*, *Comm. Math. Phys.* **202** (1999) 571 [hep-th/9803216].

[18] M.K. Prasad, *Equivalence of Eguchi-Hanson metric to two-centre Gibbons-Hawking metric*, *Phys. Lett.* **B 83** (1979) 310.

[19] A. Opfermann, Cambridge University Ph.D. thesis, 1999.

[20] B.S. Acharya, J.M. Figueroa-O'Farill and B. Spence, *Branes at angles and calibrated geometry*, *J. High Energy Phys.* **04** (1998) 012 [hep-th/9803260].

[21] E. Bergshoeff, J. Gomis and P.K. Townsend, *M-brane intersections from worldvolume superalgebras*, *Phys. Lett.* **B 421** (1998) 109 [hep-th/9711043].

[22] C. Costa and G. Papadopoulos, *Superstring dualities and p-brane bound states*, *Nucl. Phys.* **B 510** (1998) 217 [hep-th/9612204].

[23] J.H. Schwarz, *An Sl(2,$\mathbb{Z}$) multiplet of Type-IIB superstrings*, *Phys. Lett.* **B 360** (1995) 13, erratum *ibid.* **B364** (1995) 252 [hep-th/9508143].

[24] S.F. Hewson and N.D. Lambert, *Potentials and T-duality*, *Phys. Lett.* **B 408** (1997) 201 [hep-th/9703143].